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1987 J. Phys. A: Math. Gen. 20 1095

(http://iopscience.iop.org/0305-4470/20/5/021)

J. Phys. A: Math. Gen. 20 (1987) 1095-1106. Printed in the UK

Fresnel's reflection and transmission operators for stratified gyroanisotropic media

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Received 5 March 1986

Abstract. For the general case of an inhomogeneous anisotropic and gyrotropic medium a differential tensor equation, expressing the evolution of the tangential component of the field vectors of an electromagnetic wave is obtained. A fundamental solution of this equation is given by a multiplicative integral. A plane-stratified system of anisotropic and gyrotropic layers is considered. By means of the characteristic matrix of such a medium Fresnel's reflection and transmission operators are derived. These operators have wide utility because they describe exactly the interaction of light with any plane-stratified gyroanisotropic structure. The conservation of the normal component of the Poynting vector in such a structure allows us to find a correlation between the operators of reflection and transmission. The operator dispersion equation of the multilayer gyroanisotropic waveguide is presented. All the calculations in this paper are based on the direct manipulation of tensors and their invariants, eliminating the use of coordinate systems. This facilitates solutions and provides results of great generality which are suitable for computer use.

1. Introduction

At present the theory of propagation of waves and particles in linear and non-linear multicomponent channels, containing active and passive anisotropic and gyrotropic elements, is attracting attention. New analytical methods of description are being developed, and on their basis new numerical algorithms are being drawn up.

Multilayer anisotropic and gyrotropic structures arouse great interest [1-11]. One has to deal with such structures both under natural and laboratory conditions (ionosphere, heterolasers, narrow band filters, liquid crystals, Langmuir films, etc). Due to the one-dimensional stratification one may obtain a theoretical description of the optical properties. Different methods [1-13] are applied in the theory of propagation of waves in multilayer media, impedance methods being particularly developed. For dissipative systems, surface impedances independent of the incident waves are widely used, simplifying investigations of boundary value problems. If the medium is not dissipative, then the technique of such impedances cannot be used. In particular, it may not be used in the determination of the reflection of plane waves, incident upon the stratified non-absorbing gyroanisotropic structures.

It is this question which is considered in detail in the present paper. It could also be a starting point for consideration of the same problem in other more complicated cases (non-planar strata, non-planar waves, small non-linearities, small inhomogeneities in directions perpendicular to the direction of stratification).

The boundary problem is strictly formulated by means of surface impedance and normal refraction operators [14-17] which depend on the characteristics of the incident

0305-4470/87/051095+12$02.50

waves. The latter operators describe the space evolution of the field vectors in the direction of stratification. They generalise scalar effective indices of refraction which apply in the optics of stratified isotropic media [1, 2, 13]. The operators used here allow us to convert the problem of oblique incidence of waves to the simple problem of normal incidence. The purpose of this paper is to extend operator methods of solving boundary value problems of optics [17–19] to gyrotropic anisotropic multilayer systems.

2. Basic definitions and notation

Here we will follow an intrinsic form of notation. An operator (in the present case these are tensors of second rank) will be denoted by a single letter, for example α, and its components in the cartesian basis by α_{ij}, where i, j have values 1, 2, 3. The notation $\alpha\beta$ indicates the scalar product of tensors α and β (a tensor with components $(\alpha\beta)_{ij} = \alpha_{ik}\beta_{kj}$, where according to the repeated index conventions summation from 1-3 is understood); the action of an operator α on vector $\boldsymbol{a}$ will be written as follows: $\boldsymbol{b} = \alpha\boldsymbol{a}$ (in components $b_i = \alpha_{ij}a_j$); the scalar product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ we shall denote $\boldsymbol{ab}$.

From an operator α one can derive a complex conjugate operator α^*, a transposed operator $\tilde{\alpha}$, a Hermitian transposed operator $\alpha^+(\alpha^+ = \tilde{\alpha}^*)$, an inverse operator $\alpha^{-1}(\alpha\alpha^{-1} = 1$, where 1 is a unit tensor with components $1_{ij} = \delta_{ij}$, δ_{ij} is the Kronecker delta) and also an adjoint operator $\bar{\alpha}(\alpha\bar{\alpha} = |\alpha|$, $|\alpha|$ is the determinant of α), the components of which are given by $\bar{\alpha}_{ij} = \frac{1}{2}e_{klj}e_{mni}\alpha_{km}\alpha_{ln}$, where e_{ijk} is Levi-Civita's pseudotensor. Here we imply that $|\alpha|$ is multiplied by a unit tensor 1, which we will drop from now on. Beside its determinant, each tensor has two more invariants: the trace of the tensor α_t $(\alpha_t = \alpha_{ii})$ and the trace of the adjoint tensor $\bar{\alpha}_t$.

The simplest operator is the dyad $\alpha = \boldsymbol{a} \cdot \boldsymbol{b}$, equal to the tensor product of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$. The components of the dyad are given by $(\boldsymbol{a} \cdot \boldsymbol{b})_{ij} = a_i b_j$. Some properties of the dyad are represented by the following relations:

$$\tilde{\alpha} = \boldsymbol{b} \cdot \boldsymbol{a} \qquad \alpha_t = \boldsymbol{ab} \qquad |\alpha| = 0 \qquad \bar{\alpha} = 0 \qquad \alpha\boldsymbol{c} = (\boldsymbol{bc})\boldsymbol{a}. \tag{1}$$

To any non-zero vector $\boldsymbol{a}$ in three-dimensional space there is an antisymmetric tensor of second rank $\boldsymbol{a}^{\times}$ (called the dual of $\boldsymbol{a}$) with components $(\boldsymbol{a}^{\times})_{ik} = e_{ijk}a_j$. The vector product $[\boldsymbol{ab}]$ of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ may be written as follows:

$$[\boldsymbol{ab}] = \boldsymbol{a}^{\times}\boldsymbol{b} = \boldsymbol{ab}^{\times}. \tag{2}$$

The contraction of tensors $\boldsymbol{a}^{\times}$ and $\boldsymbol{b}^{\times}$ is the tensor

$$\boldsymbol{a}^{\times}\boldsymbol{b}^{\times} = \boldsymbol{b} \cdot \boldsymbol{a} - \boldsymbol{ba} = \boldsymbol{b} \cdot \boldsymbol{a} - (\boldsymbol{ba})1. \tag{3}$$

Making use of the previous definition we obtain the useful relations

$$\tilde{\boldsymbol{a}}^{\times} = -\boldsymbol{a}^{\times} \qquad \bar{\boldsymbol{a}}^{\times} = \boldsymbol{a} \cdot \boldsymbol{a} \qquad (\boldsymbol{a}^{\times})_t = 0 \qquad |\boldsymbol{a}^{\times}| = 0. \tag{4}$$

Planar tensors are widely used here, the properties of which are given in appendix 1.

The rules of coordinate-free writing of tensors and different operations with them are described in detail in [20, 21]. Such a form of writing enables us to avoid the cumbersome calculations required by the usual component techniques; furthermore, results are obtained in a form suitable for programming. This fact simplifies the process of numerical calculation.

3. Characteristic matrix operator of a stratified medium

The tensor fields of the electromagnetic constants of stratified media are described by means of permittivity tensors ε and permeability tensors μ and two gyration pseudotensors (α and β). For every uniform stratum, the thickness of which is more than the wavelength, these tensors can always be introduced. They characterise the connection between vectors of the electromagnetic field at frequency ω [21-23]:

$$\boldsymbol{D} = \varepsilon \boldsymbol{E} + \mathrm{i}\alpha \boldsymbol{H} \qquad \boldsymbol{B} = \mathrm{i}\beta \boldsymbol{E} + \mu \boldsymbol{H}. \tag{5}$$

Here $\boldsymbol{E}$, $\boldsymbol{D}$, $\boldsymbol{H}$ and $\boldsymbol{B}$ are complex vectors of strength and induction of the electric and magnetic fields. If ε, μ, α and β are complex non-symmetric tensors, then equations (5) describe an absorbing anisotropic and gyrotropic medium, subjected to the influence of external electric and magnetic fields and elastic deformations. In non-absorbing media the relations

$$\varepsilon = \varepsilon^{+} \qquad \beta = -\alpha^{+} \qquad \mu = \mu^{+} \tag{6}$$

hold true [21]. If a crystal does not possess a magnetic structure and there are no external magnetic fields, then material parameters satisfy the following conditions [21]:

$$\varepsilon = \tilde{\varepsilon} \qquad \beta = -\tilde{\alpha} \qquad \mu = \tilde{\mu}. \tag{7}$$

A plane-stratified anisotropic medium is characterised by tensor functions

$$\varepsilon = \varepsilon(z) \qquad \mu = \mu(z) \qquad \alpha = \alpha(z) \qquad \beta = \beta(z) \tag{8}$$

where $z = \boldsymbol{qr}$, $\boldsymbol{q}$ being a unit real vector normal to the strata. When the wave $\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}_0 \exp[\mathrm{i}(k\boldsymbol{mr} - \omega t)]$ is incident upon the interface of the stratified medium, a refracted wave arises, described by

$$\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{E}(z) \exp[\mathrm{i}(k\boldsymbol{br} - \omega t)] \tag{9}$$

where $k = \omega/c$ and $\boldsymbol{b}$ is a tangential component of the refraction vector $\boldsymbol{m}(\boldsymbol{b} = I\boldsymbol{m}, \boldsymbol{m} = \boldsymbol{k}'/k)$. Here $\boldsymbol{k}'$ is a wavevector and $I = -\boldsymbol{q}^{\times 2}$ is a projection operator onto the plane normal to the vector $\boldsymbol{q}$. For the wave $\boldsymbol{E}(\boldsymbol{r}, t)$ (9) Maxwell's equations reduce to the form

$$(\boldsymbol{q}^{\times}\mathrm{d}/\mathrm{d}z + \mathrm{i}k\boldsymbol{b}^{\times})\boldsymbol{H} = -\mathrm{i}k\boldsymbol{D} \qquad (\boldsymbol{q}^{\times}\mathrm{d}/\mathrm{d}z + \mathrm{i}k\boldsymbol{b}^{\times})\boldsymbol{E} = \mathrm{i}k\boldsymbol{B}. \tag{10}$$

The field vectors are coupled by the constitutive equations (5) and the additional relations $\boldsymbol{qD} = \boldsymbol{aH}$, $\boldsymbol{qB} = -\boldsymbol{aE}$ and $\boldsymbol{a} = [\boldsymbol{bq}]$ result from (10). Therefore, the vectors $\boldsymbol{H}$ and $\boldsymbol{E}$ may be expressed through their tangential components $\boldsymbol{H}_\tau = I\boldsymbol{H}$, $\boldsymbol{E}_\tau = I\boldsymbol{E}$ as follows:

$$\begin{pmatrix} \boldsymbol{H} \\ \boldsymbol{E} \end{pmatrix} = V \begin{pmatrix} \boldsymbol{H}_\tau \\ \boldsymbol{E}_\tau \end{pmatrix} \qquad V = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} + \begin{pmatrix} \boldsymbol{q} \cdot \boldsymbol{v}_1 & \boldsymbol{q} \cdot \boldsymbol{v}_2 \\ \boldsymbol{q} \cdot \boldsymbol{v}_3 & \boldsymbol{q} \cdot \boldsymbol{v}_4 \end{pmatrix} \tag{11}$$

where

$$\begin{gathered} \boldsymbol{v}_1 = Q[-\boldsymbol{q}(\varepsilon_q \mu + \beta_q \alpha) I - \mathrm{i}\beta_q \boldsymbol{a}] \qquad \boldsymbol{v}_2 = Q[-\varepsilon_q \boldsymbol{a} + \mathrm{i}\boldsymbol{q}(\beta_q \varepsilon - \varepsilon_q \beta) I] \\ \boldsymbol{v}_3 = Q[\mu_q \boldsymbol{a} + \mathrm{i}\boldsymbol{q}(\alpha_q \mu - \mu_q \alpha) I] \qquad \boldsymbol{v}_4 = Q[-\boldsymbol{q}(\mu_q \varepsilon + \alpha_q \beta) I + \mathrm{i}\alpha_q \boldsymbol{a}] \\ Q = (\varepsilon_q \mu_q + \alpha_q \beta_q)^{-1} \qquad \varepsilon_q = \boldsymbol{q}\varepsilon\boldsymbol{q} \qquad \mu_q = \boldsymbol{q}\mu\boldsymbol{q} \qquad \alpha_q = \boldsymbol{q}\alpha\boldsymbol{q} \qquad \beta_q = \boldsymbol{q}\beta\boldsymbol{q}. \end{gathered} \tag{12}$$

Taking into account relations (5) and (11) one can derive the following system from (10) [19]:

$$\frac{\mathrm{d}}{\mathrm{d}z} \begin{pmatrix} \boldsymbol{H}_\tau \\ [\boldsymbol{qE}] \end{pmatrix} = \mathrm{i}k\mathsf{M} \begin{pmatrix} \boldsymbol{H}_\tau \\ [\boldsymbol{qE}] \end{pmatrix} \qquad \mathsf{M} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{13}$$

where

$$
\begin{aligned}
A &= \mathrm{i}\boldsymbol{q}^{\times}\alpha I + \{\boldsymbol{q}^{\times}\varepsilon\boldsymbol{q}\cdot \boldsymbol{v}_3 + (\boldsymbol{b}+\mathrm{i}\boldsymbol{q}^{\times}\alpha\boldsymbol{q})\cdot \boldsymbol{v}_1\} \\
B &= -\boldsymbol{q}^{\times}\varepsilon\boldsymbol{q}^{\times} + \{\boldsymbol{q}^{\times}\varepsilon\boldsymbol{q}\cdot[\boldsymbol{q}\boldsymbol{v}_4] + (\boldsymbol{b}+\mathrm{i}\boldsymbol{q}^{\times}\alpha\boldsymbol{q})\cdot[\boldsymbol{q}\boldsymbol{v}_2]\} \\
C &= I\mu I + \{(-\boldsymbol{a}+\mathrm{i}I\beta\boldsymbol{q})\cdot \boldsymbol{v}_3 + I\mu\boldsymbol{q}\cdot \boldsymbol{v}_1\} \\
D &= -\mathrm{i}I\beta\boldsymbol{q}^{\times} + \{(-\boldsymbol{a}+\mathrm{i}I\beta\boldsymbol{q})\cdot[\boldsymbol{q}\boldsymbol{v}_4] - I\mu\boldsymbol{q}\cdot[\boldsymbol{v}_2\boldsymbol{q}]\}.
\end{aligned}
\tag{14}
$$

In media which are free of optical activity ($\alpha = \beta = 0$) formulae (11)-(14) reduce to the previously considered relations [17].

The system of equations (13) describes both homogeneous and inhomogeneous waves. In addition relations (14) distinctly show the connection between the equations of propagation of electromagnetic waves in a stratified medium with material parameters ε, μ, α and β. This simplifies the analysis of the properties of the operator coefficients A, B, C and D. For instance, considering transparent media and the real vector parameter $\boldsymbol{b}$ we obtain the following expressions from relations (14):

$$
A = D^{+} \qquad B = B^{+} \qquad C = C^{+}. \tag{15}
$$

These expressions are employed below in analysing properties of characteristic matrices of transparent stratified systems.

Consider an inhomogeneous medium, represented by a system of inhomogeneous anisotropic layers. The parameters of each of these layers, $\varepsilon(z)$, $\mu(z)$, $\alpha(z)$, $\beta(z)$, are continuous tensor functions. The general fundamental solution of equation (13) in this medium is expressed by a multiplicative integral [24]:

$$
\begin{pmatrix} \boldsymbol{H}_\tau(z) \\ [\boldsymbol{q}\boldsymbol{E}(z)] \end{pmatrix} = P \begin{pmatrix} \boldsymbol{H}_\tau(z_0) \\ [\boldsymbol{q}\boldsymbol{E}(z_0)] \end{pmatrix} \qquad P = \int_{z_0}^{z} (E + \mathrm{i}k\mathbf{M}(\xi)\,\mathrm{d}\xi) \qquad E = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \tag{16}
$$

where $\boldsymbol{H}_\tau(z_0)$ and $[\boldsymbol{q}\boldsymbol{E}(z_0)]$ are arbitrary vector parameters and P is a characteristic matrix of the stratified anisotropic system which couples the values of fields on the first and the last boundaries of the system. Some properties of the multiplicative integral P, and also of operations with block matrices, are given in appendix 3.

By calculating the multiplicative integral in the coordinate system connected with the boundary one has to deal with matrices of dimension 4×4. The calculations are essentially simplified by using the following representation of the multiplicative integral, which allows us to operate with 2×2 matrices:

$$
P = \begin{pmatrix} I & I \\ \gamma'(z) & \gamma''(z) \end{pmatrix} \begin{pmatrix} P'_H & 0 \\ 0 & P''_H \end{pmatrix} \begin{pmatrix} I & I \\ \gamma'(z) & \gamma''(z) \end{pmatrix}^{-1} \tag{17}
$$

where

$$
P'^{,''}_H = \int_{z_0}^{z} (I + \mathrm{i}kN'^{,''}_H(z)\,\mathrm{d}z)
$$

$N'^{,''}_H(z) = A(z) + B(z)\gamma'^{,''}(z)$ and $\gamma'(z)$ and $\gamma''(z)$ are tensor impedance functions, being the solutions of Riccati's tensor equation [17, 18]

$$
\frac{1}{\mathrm{i}k}\frac{\mathrm{d}\gamma}{\mathrm{d}z} + \gamma B\gamma + \gamma A - D\gamma - C = 0. \tag{18}
$$

Impedance $\gamma(z)$ connects vectors $[\boldsymbol{qE}]$ and $\boldsymbol{H}_\tau$: $[\boldsymbol{qE}]=\gamma \boldsymbol{H}_\tau$. An iteration procedure to solve equation (18) is given in appendix 2.

4. Correlation between Fresnel's reflection and transmission operators for a stratified structure

The fundamental solution (18) permits us to obtain reflection and transmission operators for electromagnetic waves by multilayer gyroanisotropic systems, and also to find the dispersion equation and the characteristics of modes of a gyroanisotropic plane-stratified waveguide.

Consider a system of $N-1$ inhomogeneous layers, in the boundaries of which the matrix M (13) is a continuous function. The characteristic matrix P (16) for the given value of the vector parameter $\boldsymbol{b}$ permits us to solve the problems of reflection of light, incident both on the first and on the last boundary of the structure. Let us find the correlation between the solutions of these two problems. Let $\boldsymbol{H}_{\tau 0}$ and $\boldsymbol{H}'_{\tau N}$ be the tangential components of the field vectors of waves incident on the first and last interfaces, respectively (see figure 1). The dependance of the field vectors of these two waves on the tangential component of the radius vector $\boldsymbol{Ir}$ is described by just the same factor $\exp(ik\boldsymbol{br})$. Then the electromagnetic field in all the layers is described by the function (9). Let each layer be characterised by material equations of the form (5). The surface impedance tensors of incident ($\boldsymbol{H}_{\tau 0}$, $\boldsymbol{H}'_{\tau N}$) and reflected ($\boldsymbol{H}'_{\tau 0}$, $\boldsymbol{H}_{\tau N}$) waves are equal to γ_0, γ'_N, γ'_0, γ_N, respectively. In practice, when the system of anisotropic layers is surrounded by an isotropic medium, all these four waves are harmonic and

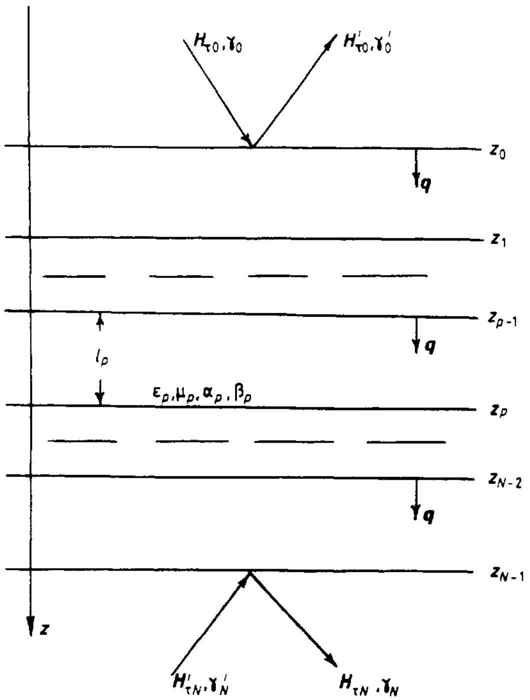

Figure 1. Geometry of the system of gyroanisotropic layers.

their impedance tensors are given by the formulae $\gamma_0 = \gamma_N = -\gamma'_0 = -\gamma'_N = (\mu_0 I - \boldsymbol{a}\cdot\boldsymbol{a}/\varepsilon_0)/(\varepsilon_0\mu_0 - \boldsymbol{b}^2)^{1/2}$, where ε_0 is the permittivity and μ_0 is the permeability of the isotropic medium. As the vectors $\boldsymbol{H}_\tau$, $[\boldsymbol{qE}]$ are continuous on the boundaries, then the field vectors on the first and the last boundaries are connected by the relation

$$\begin{pmatrix} \boldsymbol{H}_{\tau N}(z_{N-1}) + \boldsymbol{H}'_{\tau N}(z_{N-1}) \\ [\boldsymbol{q}, \boldsymbol{E}_N(z_{N-1}) + \boldsymbol{E}'_N(z_{N-1})] \end{pmatrix} = P \begin{pmatrix} \boldsymbol{H}_{\tau 0}(z_0) + \boldsymbol{H}'_{\tau 0}(z_0) \\ [\boldsymbol{q}, \boldsymbol{E}_0(z_0) + \boldsymbol{E}'_0(z_0)] \end{pmatrix}. \tag{19}$$

Using the impedance tensors γ_0, γ'_0, γ_N, γ'_N from expression (19) it is not difficult to find Fresnel's reflection and transmission operators for the system of anisotropic layers

$$\begin{aligned} \boldsymbol{H}'_{\tau 0}(z_0) &= r\boldsymbol{H}_{\tau 0}(z_0) + d'\boldsymbol{H}'_{\tau N}(z_{N-1}) \\ \boldsymbol{H}_{\tau N}(z_{N-1}) &= d\boldsymbol{H}_{\tau 0}(z_0) + r'\boldsymbol{H}'_{\tau N}(z_{N-1}) \end{aligned} \tag{20}$$

$$r = \left[(-\gamma_N, I)P\begin{pmatrix} I \\ \gamma'_0 \end{pmatrix}\right]^{-}\left[(\gamma_N, -I)P\begin{pmatrix} I \\ \gamma_0 \end{pmatrix}\right] \tag{21}$$

$$d = \left[(-\gamma'_0, I)P^{-1}\begin{pmatrix} I \\ \gamma_N \end{pmatrix}\right]^{-}(\gamma_0 - \gamma'_0) \tag{22}$$

$$r' = \left[(-\gamma'_0, I)P^{-1}\begin{pmatrix} I \\ \gamma_N \end{pmatrix}\right]^{-}\left[(\gamma'_0, -I)P^{-1}\begin{pmatrix} I \\ \gamma'_N \end{pmatrix}\right] \tag{23}$$

$$d' = \left[(\gamma_N, -I)P\begin{pmatrix} I \\ \gamma'_0 \end{pmatrix}\right]^{-}(\gamma_N - \gamma'_N). \tag{24}$$

Here r (21), d (22) (r' (23), d' (24)) are Fresnel's reflection and transmission tensors for the wave $\boldsymbol{H}_{\tau 0}(\boldsymbol{H}'_{\tau N})$ incident on the system from the region $z < z_0 (z > z_{N-1})$.

In the absence of absorption the reflection and transmission tensors (21)-(24) are connected with each other. One can determine this connection by considering the flow of energy.

For the normal component $\boldsymbol{S}_q$ of the average vector of the flow of energy we have

$$\boldsymbol{S}_q = S_q\boldsymbol{q}$$
$$S_q = \boldsymbol{Sq} = \frac{c}{16\pi}\{[\boldsymbol{qE}]\boldsymbol{H}^*_\tau + [\boldsymbol{qE}^*]\boldsymbol{H}_\tau\} = \frac{c}{16\pi}(\boldsymbol{H}^*_\tau, [\boldsymbol{qE}^*])\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{H}_\tau \\ [\boldsymbol{qE}] \end{pmatrix}. \tag{25}$$

Rewrite relations (15) in the form

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} M^+ \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \mathsf{M}. \tag{26}$$

Using formula (26) one can derive the expression

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} P^+ \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} P = E. \tag{27}$$

Comparing normal components $\boldsymbol{S}_q(z)$ and $\boldsymbol{S}_q(z_0)$ at the points z and z_0 with the help of formulae (16) and (27) we obtain

$$\boldsymbol{S}_q(z) = \boldsymbol{S}_q(z_0) \tag{28}$$

for an arbitrary point z. Substituting expressions (25), (19) and (20) into (28) for the points z_0 and z_{N-1} and taking into account the arbitrariness of the amplitudes of the incident waves, we find the following correlations between the reflection and transmission tensors (21)-(24):

$$\begin{aligned}
&(I+r)^{+}(\gamma_0+\gamma_0' r)+(\gamma_0+\gamma_0' r)^{+}(I+r)=d^{+}(\gamma_N+\overset{+}{\gamma}_N)d\\
&(I+r')^{+}(\gamma_N'+\gamma_N r')+(\gamma_N'+\gamma_N r')^{+}(I+r')=d'^{+}(\gamma_0'+\gamma_0'^{+})d'\\
&(I+r)^{+}\gamma_0' d'+(\gamma_0+\gamma_0' r)^{+}d'=d^{+}(\gamma_N'+\gamma_N r')+d^{+}\overset{+}{\gamma}_N(I+r')\\
&(I+r')^{+}\gamma_N d+(\gamma_N'+\gamma_N r')^{+}d=d'^{+}(\gamma_0+\gamma_0' r)+d'^{+}\gamma_0'^{+}(I+r).
\end{aligned} \tag{29}$$

In the particular case of normal incidence on the system which is in air the expressions (29) are essentially simplified:

$$r^{+}r+d^{+}d=r'^{+}r'+d'^{+}d'=I \qquad r^{+}d'+d^{+}r'=0. \tag{30}$$

5. Operator dispersion equation of plane-stratified waveguides

The above formulae permit us to find, in general form, dispersion equations and eigenmodes of plane-stratified continuously inhomogeneous waveguides. In the regions $z<z_0$, $z>z_{N-1}$, only the waves $\boldsymbol{H}'_{\tau0}$, $\boldsymbol{H}_{\tau N}$ exist, out of the waveguide. Substituting $\boldsymbol{H}_{\tau0}=\boldsymbol{H}'_{\tau N}=0$ into relationship (19) yields

$$(-\gamma_N,\, I)P\begin{pmatrix} I\\ \gamma_0'\end{pmatrix}\boldsymbol{H}'_{\tau0}(z_0)=0 \tag{31}$$

where P is a characteristic matrix. Equation (31) has a non-trivial solution for the vector $\boldsymbol{H}'_{\tau0}(z_0)$ only if a tensor $\varkappa=(-\gamma_N,\, I)P(\begin{smallmatrix} I\\ \gamma_0'\end{smallmatrix})$ is a dyad, i.e. $\bar{\varkappa}_t=0$ (see relation (1)). The relationship

$$\overline{(-\gamma_N,\, I)P\begin{pmatrix} I\\ \gamma_0'\end{pmatrix}_t}=0 \tag{32}$$

just represents the dispersion equation for a plane-stratified waveguide. The impedance tensors γ_0', γ_N and also the characteristic matrix P (16) depend on the vector parameter $\boldsymbol{b}$, the frequency ω and parameters ε_p, μ_p, α_p, β_p and l_p $(p=1,2,\ldots,N-1)$, where l_p is the depth of the pth layer. If the latter are given, then equation (32) defines the spectrum of the assumed values of the vector $\boldsymbol{b}$, which determines the mode structure of the waveguide.

Let vector $\boldsymbol{b}_i$ be one of the solutions of equation (32), and $\boldsymbol{h}_{\tau i}=\boldsymbol{d}\varkappa(\boldsymbol{b}_i)\boldsymbol{q}^{\times}$ be an eigenvector of the dyad $\varkappa(\boldsymbol{b}_i)$, corresponding to the zero eigenvalue $\varkappa(\boldsymbol{b}_i)\boldsymbol{h}_{\tau i}=0$, $\boldsymbol{h}_{\tau i}\boldsymbol{q}=0$. Here vector $\boldsymbol{d}$ is any vector, satisfying the condition $\boldsymbol{d}\varkappa(\boldsymbol{b}_i)\neq 0$. Knowing vector $\boldsymbol{h}_{\tau i}$ one can find values of the vectors of a single mode in any point of the waveguide:

$$\begin{pmatrix}\boldsymbol{H}_{\tau i}(z)\\ [\boldsymbol{q}\boldsymbol{E}_i(z)]\end{pmatrix}=\int_{z_0}^{z}(E+\mathrm{i}kM(\xi)\,\mathrm{d}\xi)\begin{pmatrix}\boldsymbol{h}_{\tau i}\\ \gamma_0'\boldsymbol{h}_{\tau i}\end{pmatrix}. \tag{33}$$

To every solution $\boldsymbol{b}_i$ of the dispersion equation (32) there corresponds a certain type of wave (mode of the waveguide). In the case of an isotropic waveguide equation (32) decomposes into four well known scalar dispersion equations [25] for even and odd TE and TM modes.

6. Conclusion

The general relationships obtained here may be used in many applications. For example, they may be applied in the study of electromagnetic waves in a plasma, in the ionosphere and in the theory of waveguides.

Appendix 1. Some properties of planar tensors

In the present paper planar tensors are widely used, satisfying the following conditions:

$$\boldsymbol{q}\alpha = 0 \qquad \alpha\boldsymbol{q} = 0. \tag{A1.1}$$

In the general case α is the complex non-Hermitian tensor. The projector operator I of the two-dimensional subspace orthogonal to the vector $\boldsymbol{q}$ has the form

$$I = -\boldsymbol{q}^{\times 2} = 1 - \boldsymbol{q}\cdot\boldsymbol{q} \qquad I^2 = I \qquad \boldsymbol{q}I = 0 \qquad I\boldsymbol{q} = 0 \qquad I\alpha = \alpha I = \alpha. \tag{A1.2}$$

Let us denote non-zero eigenvalues of the tensor α by λ_+ and λ_-. They are derived from the characteristic equation

$$|\alpha - \lambda_\pm| = \lambda_\pm^3 + \alpha_t\lambda_\pm^2 - \bar{\alpha}_t\lambda_\pm = 0$$

where the determinant of the sum of two tensors is found from the formula $|\alpha+\beta| = |\alpha| + |\beta| + (\bar{\alpha}\beta)_t + (\alpha\bar{\beta})_t$ [21]:

$$\lambda_\pm = \tfrac{1}{2}\{\alpha_t \pm [(\alpha_t)^2 - 4\bar{\alpha}_t]^{1/2}\}. \tag{A1.3}$$

Tensors $\boldsymbol{q}\cdot\boldsymbol{q}$ and

$$\rho_\pm = \pm(\alpha - \lambda_\mp I)/(\lambda_+ - \lambda_-) \tag{A1.4}$$

represent projection operators of eigensubspaces of tensor α, while ρ_+ and ρ_- correspond to eigenvalues λ_+ and λ_-, and $\boldsymbol{q}\cdot\boldsymbol{q}$ to a zero eigenvalue. Tensors $\rho_\pm$, as can easily be shown, are dyads, i.e. $\bar{\rho}_\pm = 0$. Projectors $\rho_\pm$ satisfy the following relations:

$$\rho_\pm^2 = \rho_\pm \qquad \rho_+ + \rho_- = I \qquad \rho_\pm\rho_\mp = 0 \qquad \boldsymbol{q}\rho_\pm = 0 \qquad \rho_\pm\boldsymbol{q} = 0 \qquad \rho_{\pm t} = 1. \tag{A1.5}$$

It is often convenient to use the spectral form of the tensor α

$$\alpha = \lambda_+\rho_+ + \lambda_-\rho_-. \tag{A1.6}$$

Let the planar tensor α not be a dyad, i.e. $\bar{\alpha} \neq 0$. Then there is a pseudoinverse operator α^-, defined by the relations

$$\alpha\alpha^- = \alpha^-\alpha = I \qquad \boldsymbol{q}\alpha^- = 0 \qquad \alpha^-\boldsymbol{q} = 0. \tag{A1.7}$$

The explicit form of this operator we find from the Cayley-Hamilton theorem [24] according to which the tensor satisfies the relationship [21]: $\bar{\alpha} - \bar{\alpha}_t = \alpha(\alpha - \alpha_t)$. For planar tensors this relationship becomes

$$\alpha^2 - \alpha_t\alpha + \bar{\alpha}_t I = 0. \tag{A1.8}$$

Hence

$$\alpha^- = (\alpha_t I - \alpha)/\bar{\alpha}_t. \tag{A1.9}$$

If $\lambda_+ \neq \lambda_-$, we have

$$\alpha^- = (1/\lambda_+)\rho_+ + (1/\lambda_-)\rho_-. \tag{A1.10}$$

Let us find a tensor x, satisfying the equation

$$x^2 = \alpha \tag{A1.11}$$

where x and α are planar tensors. Using expression (A1.8) one may show that the tensor x is equal to

$$x = \alpha^{1/2} = \frac{\sqrt{\bar{\alpha}_t} I + \alpha}{(\alpha_t + 2\sqrt{\bar{\alpha}_t})^{1/2}} = \sqrt{\lambda_+}\rho_+ + \sqrt{\lambda_-}\rho_- . \tag{A1.12}$$

Consider the exponential of the planar tensor α. The exponential $\exp\alpha$ is understood as a series

$$\exp\alpha = I + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!}$$
$$= \exp\left(\frac{\alpha_t}{2}\right)\left(\cosh[\tfrac{1}{4}(\alpha_t)^2 - \bar{\alpha}_t]^{1/2} I + \frac{\sinh[\tfrac{1}{4}(\alpha_t)^2 - \bar{\alpha}_t]^{1/2}}{[\tfrac{1}{4}(\alpha_t)^2 - \bar{\alpha}_t]^{1/2}}(\alpha - \tfrac{1}{2}\alpha_t I)\right) \tag{A1.13}$$

This series differs from the standard definition of the exponential by the first term (see, for example, [24]). However, the application of formula (A1.13) should not lead to any misunderstanding, as in our case the operator (A1.13) acts on the vectors lying in the subspace defined by tensor I. This subspace (plane) is invariant to the action of the operator $\exp\alpha$. Therefore, one can ignore the fact that the exponential $\exp\alpha$ may be defined out of this plane. Substituting the spectral form of the tensor α (A1.6) in the relation (A1.13) yields

$$\exp\alpha = e^{\lambda_+}\rho_+ + e^{\lambda_-}\rho_- . \tag{A1.14}$$

Appendix 2. Iterative methods of solving Riccati's equation for impedance tensors

Riccati's equation (18) has no solutions independent of the coordinate z. If the tensors A, B, C and D (14) are only slightly changed at distances of about $1/k$, then one can look for solutions of (18) in the form of a series:

$$\gamma(z) = \gamma_0(z) + \sum_{n=1}^{\infty} \gamma_n(z)/(\mathrm{i}k)^n \tag{A2.1}$$

where $\gamma_0(z)$ is a solution of the algebraic Riccati equation with tensor coefficients dependent on z

$$\gamma_0 B \gamma_0 + \gamma_0 A - D\gamma_0 - C = 0. \tag{A2.2}$$

Substituting (A2.1) in equation (18), we obtain the recurrence relations

$$\begin{aligned} &\gamma_1(B\gamma_0 + A) + (\gamma_0 B - D)\gamma_1 + \mathrm{d}\gamma_0/\mathrm{d}z = 0 \\ &\gamma_n(B\gamma_0 + A) + (\gamma_0 B - D)\gamma_n + \mathrm{d}\gamma_{n-1}/\mathrm{d}z + \sum_{S=1}^{n-1} \gamma_S B \gamma_{n-S} = 0 \qquad n = 2, 3, \ldots \end{aligned} \tag{A2.3}$$

from which one can derive the tensor terms of the series (A2.1).

Let us find the tensor γ_0. As the degree of anisotropy of the tensors ε and μ is usually not great and the parameters of gyration α and β are small, then the norms of tensors A and D are much less than the norms of the tensors B and C. Therefore, consider first the solution of the equation

$$\gamma_0^{(0)} B \gamma_0^{(0)} = C. \tag{A2.4}$$

Multiplying (A2.4) by the tensor B and using the idea of the pseudoinverse operator (A1.9) and the square root from the planar tensor (A1.13) we obtain

$$\gamma_0^{(0)} = \frac{C - (\bar{C}_t/\bar{B}_t)^{1/2} \boldsymbol{q}^{\times} \tilde{B} \boldsymbol{q}^{\times}}{[(BC)_t + 2(\bar{B}_t \bar{C}_t)^{1/2}]^{1/2}}. \tag{A2.5}$$

The solution of equation (A2.2) in the general case, taking into consideration the small values of the norms of tensors A and D in comparison with the norms of tensors B and C, can be found by means of the iterative formula

$$\gamma_0^{(k+1)} = B^{-}[B(C + D\gamma_0^{(k)} - \gamma_0^{(k)} A)]^{1/2} \tag{A2.6}$$

where $k = 0, 1, 2, \ldots$, and as the first approximation to $\gamma_0^{(0)}$ we use expression (A2.5).

Appendix 3. Block matrices

In the present paper block matrices of dimension 4×4 are widely used. These are the matrices, the elements of which are planar tensors $A_1, B_1, \ldots, D_2$, satisfying the conditions (A1.1)

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \qquad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}. \tag{A3.1}$$

The sum and the product of such matrices are determined by the relations

$$M_1 + M_2 = \begin{pmatrix} A_1 + A_2 & B_1 + B_2 \\ C_1 + C_2 & D_1 + D_2 \end{pmatrix} \qquad M_1 M_2 = \begin{pmatrix} A_1 A_2 + B_1 C_2 & A_1 B_2 + B_1 D_2 \\ C_1 A_2 + D_1 C_2 & C_1 B_2 + D_1 D_2 \end{pmatrix} \tag{A3.2}$$

For block matrices such as (A3.1) an operation of pseudoinversion may be introduced. The pseudoinverse matrix M^{-} satisfies the following relationships:

$$M^{-}M = MM^{-} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \qquad \begin{pmatrix} \boldsymbol{q}\cdot\boldsymbol{q} & 0 \\ 0 & \boldsymbol{q}\cdot\boldsymbol{q} \end{pmatrix} M^{-} = 0 \qquad M^{-} \begin{pmatrix} \boldsymbol{q}\cdot\boldsymbol{q} & 0 \\ 0 & \boldsymbol{q}\cdot\boldsymbol{q} \end{pmatrix} = 0. \tag{A3.3}$$

In this paper we have multiplied different block matrices. For example,

$$(\gamma_{N}, -I) P \begin{pmatrix} I \\ \gamma_0 \end{pmatrix} = (\gamma_{N}, -I) \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} I \\ \gamma_0 \end{pmatrix}$$

is a multiplication of a 2×4 matrix by 4×4 and 4×2 matrices.

The multiplicative integral of the block-matrix function is determined as follows. Divide the interval (z_0, z) into n parts, introducing intermediate points $z_1, z_2, \ldots, z_{n-1}$, $\Delta z_k = z_k - z_{k-1}$ $(k = 1, 2, \ldots, n)$, $z_n = z$. Then the multiplicative integral is the following expression [24]:

$$P = \int_{z_0}^{z} (E + M\, \mathrm{d}\zeta) = \lim_{\Delta z_k \to 0} [E + M(z_n)\Delta z_n] \ldots [E + M(z_1)\Delta z_1]. \tag{A3.4}$$

Some properties of the multiplicative integral are given below:

$$\int_{z_0}^{z}(E+M\,\mathrm{d}\zeta)=\int_{z_1}^{z}(E+M\,\mathrm{d}\zeta)\int_{z_0}^{z_1}(E+M\,\mathrm{d}\zeta) \tag{A3.5}$$

$$\left(\int_{z_0}^{z}(E+M\,\mathrm{d}\zeta)\right)^{-1}=\int_{z}^{z_0}(E+M\,\mathrm{d}\zeta) \tag{A3.6}$$

$$\begin{gathered}\int_{z_0}^{z}[E+(M_1+D_\zeta M_2)\,\mathrm{d}\zeta]=M_2(z)\int_{z_0}^{z}(E+M_2^{-}M_1M_2\,\mathrm{d}\zeta)M_2^{-}(z_0)\\ D_\zeta M=(\mathrm{d}M/\mathrm{d}\zeta)M^{-}.\end{gathered} \tag{A3.7}$$

Using (A3.5) it can be easily shown that for the system of $N-1$ layers the matrix P may be represented as

$$P=P_{N-1}P_{N-2}\dots P_1 \qquad P_p=\int_{z_{p-1}}^{z_p}(E+M(\zeta)\,\mathrm{d}\zeta) \qquad p=1,2,\dots,N-1 \tag{A3.8}$$

where P_p is the multiplicative integral for the pth layer. The pseudoinverse matrix (A3.6) has the form

$$P^{-1}=P_1^{-1}P_2^{-1}\dots P_{N-1}^{-1}. \tag{A3.9}$$

If, within the limits of a certain layer, the values of the function $M(\zeta)$ at two arbitrary points ζ_1 and ζ_2 commute with themselves, i.e. $M(\zeta_1)M(\zeta_2)=M(\zeta_2)M(\zeta_1)$, $\zeta_{1,2}\in[z_{p-1},z_p]$, then the matrix P of this layer becomes

$$P_p=\exp\left(\int_{z_{p-1}}^{z_p}M(\zeta)\,\mathrm{d}\zeta\right). \tag{A3.10}$$

For a homogeneous medium the proper multiplicative integral reduces to the exponential (A1.13)

$$P=\exp(lM) \tag{A3.11}$$

where l is the interval in the medium.

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